Indian Statistical Institute, Bangalore
B. Math.
First Year, First Semester
Analysis I

Mid-term examination — Date : Sept. 6, 2013
Total Marks: 110 Maximum marks: 100 — Time: 3 hours

1. Let A, B, C be non-empty sets and let $f : A \to B$; $g : B \to C$ and $h : A \to C$ be functions with $h = g \circ f$ (composition). Prove or disprove the following:

   (i) If f, g are surjective then h is surjective.

   (ii) If h is surjective then f, g are surjective.

   (iii) If h is injective then f, g are injective.

   [15]

2. Let $X = \{1, 2\}$ and let M be the set of all integer valued functions on X. Show that M is countable. [15]

3. Show that given any real number x there exists a natural number n such that $n > x$.

   [15]

4. Suppose $u : [0, 1] \to I\!\!R$ is a continuous function. Define a function $v : [0, 2] \to I\!\!R$ by

$$v(t) = \begin{cases} u(t) & \text{if } 0 \le t \le 1; \\ u(2-t) & \text{if } 1 < t \le 2 \end{cases}$$

   Show that v is continuous. [15]

5. Let $\{b_n\}_{n\ge1}$ and $\{c_n\}_{n\ge1}$ be two sequences of real numbers. Define a new sequence $\{a\}_{n\ge1}$ by

$$a_n = \begin{cases} b_n & \text{if } n \text{ is odd} \\ c_n & \text{if } n \text{ is even} \end{cases}$$

   Show that if $\{b_n\}, \{c_n\}$ are convergent with same limit then $\{a_n\}$ is convergent. However, the converse is not true. [20]

6. Suppose $g : [0, 1] \to [3, 5]$ is a continuous bijection. Show that either g is strictly increasing with $g(0) = 3, g(1) = 5$ or g is strictly decreasing with $g(0) = 5, g(1) = 3$. Give an example of a discontinuous bijection from $[0, 1]$ to $[3, 5]$ which is not monotonic. [20]

7. Find lim inf and lim sup of the following sequences (you should also prove your claims):

   (i) $\{x_n\}_{n\ge1}$, with $x_n = (-\frac{1}{2})^n + \frac{1}{n}$ for $n \ge 1$.

   (ii) $\{y\}_{n\ge1}$, with $y_n = (-1)^n \frac{2n^2+3n+1}{5n^2+4}$ for $n \ge 1$. [10]